3.3 The Factor Theorem

Example 1: Find the remainder when $x^3 + 2x^2 - 5x - 6$ by $x + 1$.

i) Using synthetic division:

1	1	2	-5	-6
–	↓	1	1	-6
x	1	1	-6	0

Remainder of zero!

ii) Using the Remainder Theorem:

Find $P(-1)$

$$P(-1) = (-1)^3 + 2(-1)^2 - 5(-1) - 6$$
$$= -1 + 2 + 5 - 6$$
$$P(-1) = 0$$

remainder!

A. The Factor Theorem

The Factor Theorem states that $x - a$ is a factor of a polynomial $P(x)$, if and only if, $P(a) = 0$.

So, in Example 1, $x+1$ is a factor of $x^3 + 2x^2 - 5x - 6$

Example 2: Determine if $x - 5$ is a factor of $P(x) = x^3 - 2x^2 - 33x + 90$

Use the Remainder Theorem:

$$P(5) = (5)^3 - 2(5)^2 - 33(5) + 90$$
$$= 125 - 50 - 165 + 90$$
$$= 0$$

remainder

Since $P(5) = 0$, we can say $x-5$ is a factor of $P(x)$.

Example 3: Determine if $x + 3$ is a factor of $P(x) = x^3 - 3x^2 - x + 3$.

$$P(-3) = (-3)^3 - 3(-3)^2 - (-3) + 3$$
$$= -27 - 27 + 3 + 3$$
$$= -48$$

↳ not zero!

So, $x+3$ is not a factor of $P(x)$.

B. Possible factors of a polynomial: In order to factor a polynomial, we need to find the values of "a" such that $P(a) = 0$.

C. Intergral Zero Theorem

If $x - a$ is a factor of $P(x)$ with integral coefficients, then "a" is a factor of the constant term of $P(x)$.

(integers)

Example 4: List all the possible values of "a" such that $x - a$ is a factor of $P(x) = x^3 + 3x^2 - 2x + 12.$

factors of $+12$ $\{\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12\}$

possible values of "a"

Example 5: Factor $P(x) = x^3 - 6x^2 + 7x + 6$.

① list all possible values of "a" $\{\pm 1, \pm 2, \pm 3, \pm 6\}$ factors of 6

② Use Remainder Theorem to find a value for "a" such that $P(a) = 0$

try $a = 3$

$$P(3) = (3)^3 - 6(3)^2 + 7(3) + 6$$
$$= 27 - 54 + 21 + 6$$
$$= 0$$

Only show work for the one that works. (do others on calc.)

▸ So, $x-3$ is a factor.

③ Use synthetic division to find a quotient.

-3	1	-6	7	6
–	↓	-3	9	6
x	1	-3	-2	0

$\underbrace{1 \quad -3 \quad -2}_{Q(x)}$

$$P(x) = Q(x)(x-a) + R$$

$$P(x) = (x^2 - 3x - 2)(x - 3)$$

④ Factor $Q(x)$ further, if possible.

$Q(x)$ cannot be factored.

So, $(x^2 - 3x - 2)(x - 3)$ is the final answer in factored form.

Example 6: Factor $P(x) = 2x^3 - 15x^2 + 27x - 10$.

possible values of -10 $\{\pm 1, \pm 2, \pm 5, \pm 10\}$ factors of -10

try $a = 2$

$$P(2) = 2(2)^3 - 15(2)^2 + 27(2) - 10$$
$$= 16 - 60 + 54 - 10$$
$$= 0$$

So, $(x-2)$ is a factor.

-2	2	-15	27	-10
$-$	$\downarrow$	-4	22	-10
$\times$	2	-11	5	0

$$\underbrace{2 \quad -11 \quad 5}_{Q(x)} \qquad \underbrace{0}_{R}$$

$$Q(x) = 2x^2 - 11x + 5$$

$$P(x) = \underbrace{(2x^2 - 11x + 5)}_{\text{continue factoring}}(x-2)$$

continue factoring

factor by decomposition

$$\underbrace{2x^2 - x}\ \underbrace{- 10x + 5}$$

$\underline{-1} \times \underline{-10} = 10$

$\underline{-1} + \underline{-10} = -11$

$$x(2x-1) - 5(2x-1)$$

$$(2x-1)(x-5)$$

So, $P(x) = (2x-1)(x-5)(x-2)$

Practice: p.133 #2ab, 3cd, 4ef, 5acd, 6bc, 7ad, 15