6.1 Reciprocal, Quotient and Pythagorean Identities

A. Trig Identity : A trig equation that is true for all permissible values of the variable on both sides of the equation.

Example 1: Verify the identify for $x = \frac{\pi}{3}$

$$(\tan\theta - 1)^2 = \sec^2\theta - 2\tan\theta$$

sub $x = \frac{\pi}{3}$ into the identity

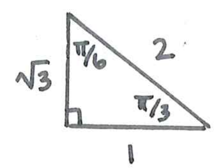
$$\left(\tan\frac{\pi}{3} - 1\right)^2 = \sec^2\frac{\pi}{3} - 2\tan\frac{\pi}{3}$$

$$(\sqrt{3} - 1)^2 = \left(\sec\frac{\pi}{3}\right)^2 - 2(\sqrt{3})$$

$$(\sqrt{3} - 1)(\sqrt{3} - 1) = (2)^2 - 2\sqrt{3}$$

$$3 - \sqrt{3} - \sqrt{3} + 1 = 4 - 2\sqrt{3}$$

$$4 - 2\sqrt{3} = 4 - 2\sqrt{3}$$



$\tan\theta = \frac{y}{x}$

$\sec\theta = \frac{r}{x}$

$\tan\frac{\pi}{3} = \frac{\sqrt{3}}{1} = \sqrt{3}$

$\sec\frac{\pi}{3} = \frac{2}{1} = 2$

left side = right side

therefore, the identity has been verified

B. Reciprocal Identities

$\csc\theta = \frac{1}{\sin\theta}$ $\quad$ $\sec\theta = \frac{1}{\cos\theta}$ $\quad$ $\cot\theta = \frac{1}{\tan\theta}$

C. Quotient Identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad\qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

Example 2: Simplify the identity

Do not move terms from one side to the other when simplifying.

a) $\sin\theta \underbrace{\cot\theta} = \cos\theta$

rewrite quotient identity

$$\cancel{\sin\theta} \cdot \frac{\cos\theta}{\cancel{\sin\theta}} = \cos\theta$$

$$\cos\theta = \cos\theta \quad \checkmark$$

b) $\dfrac{\cot\theta}{\csc\theta\cos\theta}$ rewrite

$$= \frac{\frac{\cos\theta}{\sin\theta}}{\frac{1}{\sin\theta}\cdot\cos\theta}$$

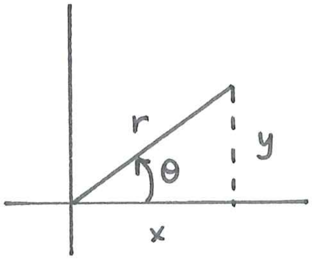
$$= \frac{\frac{\cos\theta}{\sin\theta}}{\frac{\cos\theta}{\sin\theta}} \quad \begin{matrix}\} \text{ numerator} \\ \} \text{ denominator}\end{matrix} \quad \text{same!}$$

this complex fraction can be rewritten as a multiplication

$$= \frac{\cancel{\cos\theta}}{\cancel{\sin\theta}} \cdot \frac{\cancel{\sin\theta}}{\cancel{\cos\theta}}$$

$$= 1 \quad \text{(final answer)}$$

D. Pythagorean Identities



recall: $\sin\theta = \frac{y}{r}$ $\quad \cos\theta = \frac{x}{r}$ $\quad \tan\theta = \frac{y}{x}$

$\csc\theta = \frac{r}{y}$ $\quad \sec\theta = \frac{r}{x}$ $\quad \cot\theta = \frac{x}{y}$

$$x^2 + y^2 = r^2$$

①

$$\frac{x^2}{r^2} + \frac{y^2}{r^2} = \frac{r^2}{r^2}$$

$$\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = 1$$

$$\cos^2\theta + \sin^2\theta = 1$$

②

$$\frac{x^2}{y^2} + \frac{y^2}{y^2} = \frac{r^2}{y^2}$$

$$\left(\frac{x}{y}\right)^2 + 1 = \left(\frac{r}{y}\right)^2$$

$$\cot^2\theta + 1 = \csc^2\theta$$

③

$$\frac{x^2}{x^2} + \frac{y^2}{x^2} = \frac{r^2}{x^2}$$

$$1 + \left(\frac{y}{x}\right)^2 = \left(\frac{r}{x}\right)^2$$

$$1 + \tan^2\theta = \sec^2\theta$$

Example 4: Simplify each expression as a single trig function.

a) $\sin\theta\,\underbrace{(\sin^2\theta + \cos^2\theta)}_{\text{rewrite}}\,\underbrace{\sec\theta}_{\text{rewrite}}$

(Pythagorean identity)

$= \sin\theta(1)\left(\frac{1}{\cos\theta}\right)$

$= \frac{\sin\theta}{\cos\theta}$ } rewrite as a single trig ratio

$= \tan\theta.$

rewrite

b) $\dfrac{\tan\theta(\sin^2\theta+\cos^2\theta)}{\sec\theta}$

$= \dfrac{\tan\theta\ (1)}{\sec\theta}$ rewrite

$= \dfrac{\frac{\sin\theta}{\cos\theta}}{\frac{1}{\cos\theta}}$ rewrite as a multiplication

$= \dfrac{\sin\theta}{\cancel{\cos}\theta} \cdot \dfrac{\cancel{\cos}\theta}{1}$ simplify

$= \sin\theta$

rewrite

c) $(\tan\theta - 1)^2 + 2\sin\theta\sec\theta$

$= \underbrace{(\tan\theta - 1)(\tan\theta - 1)}_{\text{Expand (FOIL)}} + 2\sin\theta\underbrace{\sec\theta}_{\text{rewrite}}$

$= \tan^2\theta - \tan\theta - \tan\theta + 1 + 2\sin\theta\left(\frac{1}{\cos\theta}\right)$

$= \underline{\tan^2\theta} - 2\tan\theta \underline{+1} + 2\left(\frac{\sin\theta}{\cos\theta}\right)$ rewrite

Pythagorean Identity

$= \sec^2\theta - 2\cancel{\tan}\theta + 2\cancel{\tan}\theta$

$= \sec^2\theta$

Practice: p.296 #3, 4, 5a, 6c, 10, 12b, 14