6.2 Sum, Difference, and Double-Angle Identities

A. Sum Identities

$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

B. Difference Identities

$$\sin(A-B) = \sin A \cos B - \cos A \sin B$$

$$\cos(A-B) = \cos A \cos B + \sin A \sin B$$

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

Example 1: Simplify the expression as a single trig function.

$$\sin\frac{\pi}{8}\cos\frac{3\pi}{5} + \cos\frac{\pi}{8}\sin\frac{3\pi}{5}$$

① Match the pattern with the corresponding identity

$A = \frac{\pi}{8}$ $\quad$ $B = \frac{3\pi}{5}$ $\quad \rightarrow$ addition identity for sine!

$\sin(A+B) = \sin\left(\frac{\pi}{8} + \frac{3\pi}{5}\right)$

② Simplify the identity

$= \sin\left(\frac{5\pi}{40} + \frac{24\pi}{40}\right)$

$= \sin\left(\frac{29\pi}{40}\right)$

Example 2: Find an <u>exact value</u> for $\cos\frac{\pi}{12}$

(no calc. !)

① Rewrite $\frac{\pi}{12}$ as the sum/difference of 2 special angles

$$\frac{4\pi}{12} - \frac{3\pi}{12} = \frac{\pi}{12} \qquad \text{or} \qquad \frac{3\pi}{12} - \frac{2\pi}{12} = \frac{\pi}{12}$$

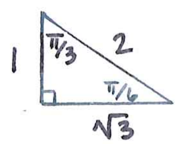
$$\underbrace{\overset{A}{\frac{\pi}{3}} - \overset{B}{\frac{\pi}{4}}}_{\text{we'll use this}} \qquad\qquad \frac{\pi}{4} - \frac{\pi}{6}$$

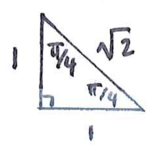
② Find a corresponding identity

$$\cos(A-B) = \cos A\cos B + \sin A \sin B$$

$$\cos\left(\frac{\pi}{12}\right) = \cos\frac{\pi}{3}\cos\frac{\pi}{4} + \sin\frac{\pi}{3}\sin\frac{\pi}{4}$$

③ Evaluate using special triangles





$\frac{\pi}{12}$ is not a special angle.

$$\frac{\pi}{3} = \frac{4\pi}{12} \ ; \quad \frac{\pi}{4} = \frac{3\pi}{12} \ ; \quad \frac{\pi}{6} = \frac{2\pi}{12}$$

step 3 cont...

$$\cos\left(\frac{\pi}{12}\right) = \left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{2}}\right) + \left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{\sqrt{2}}\right)$$

$$= \frac{1}{2\sqrt{2}} + \frac{\sqrt{3}}{2\sqrt{2}}$$

$$= \frac{1+\sqrt{3}}{2\sqrt{2}}$$

④ Rationalize the denominator

$$= \frac{1+\sqrt{3}}{2\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}}$$

$$= \frac{\sqrt{2}+\sqrt{6}}{2\cdot 2}$$

$$= \frac{\sqrt{2}+\sqrt{6}}{4}$$

Example 3: Simplify and create a double-angle identify

a) $\sin 2A = \sin(A+A)$

$$= \sin A\cos A + \sin A \cos A$$

$$= 2\sin A\cos A$$

b) $\tan 2A = \dfrac{\tan A + \tan A}{1 - \tan A\tan A}$

$$= \frac{2\tan A}{1-\tan^2 A}$$

b) $\cos 2A = \cos A\cos A - \sin A\sin A$

$$= \cos^2 A - \sin^2 A$$

or $= \cos^2 A - (1 - \cos^2 A)$

$$= 2\cos^2 A - 1$$

or $= (1-\sin^2 A) - \sin^2 A$

using Pythagorean identity

$$= 1 - 2\sin^2 A$$

D. Double Angle Identities

$$\sin 2A = 2\sin A\cos A$$

$$\cos 2A = \begin{cases} \cos^2 A - \sin^2 A \\ 2\cos^2 A - 1 \\ 1 - 2\sin^2 A \end{cases}$$

$$\tan 2A = \frac{2\tan A}{1-\tan^2 A}$$

Example 3: Rewrite as a single trig function

$$\frac{2\tan 20°}{1-\tan^2 20°}$$

Find the corresponding identity.

$\tan 2A = \dfrac{2\tan A}{1-\tan^2 A}$ $\quad A = 20°$

so... $\dfrac{2\tan 20°}{1-\tan^2 20°} = \tan 2(20°) = \tan 40°$

Example 4: Evaluate. Determine the exact value without using a calculator

$$\left.10\cos^2\left(\frac{\pi}{12}\right) - 5 \right\} \text{Factor, GCF}$$

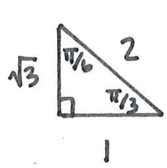
① Find a corresponding identity

$= 5\left(2\cos^2\left(\frac{\pi}{12}\right) - 1\right)$ $\quad 2\cos^2 A - 1 = \cos 2A$, $\quad A = \frac{\pi}{12}$

$= 5\left(\cos 2\left(\frac{\pi}{12}\right)\right)$

$= 5\left(\cos\frac{\pi}{6}\right)$

② Evaluate using a special Δ



$5\left(\cos\frac{\pi}{6}\right) = 5\left(\frac{\sqrt{3}}{2}\right)$

$= \dfrac{5\sqrt{3}}{2}$

Example 5: Evaluate. Determine the exact value without using a calculator

$$\left(\sin\frac{\pi}{8}\right)\left(\cos\frac{\pi}{8}\right)$$

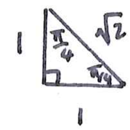
since $\sin 2A = 2\sin A\cos A$, we can say that...

$$\frac{1}{2}\sin 2A = \sin A\cos A$$

so... $\sin\frac{\pi}{8}\cos\frac{\pi}{8} = \frac{1}{2}\sin 2\left(\frac{\pi}{8}\right)$

$$= \frac{1}{2}\sin\left(\frac{\pi}{4}\right)$$

evaluate with special $\triangle$

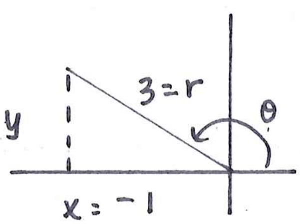


$$= \frac{1}{2}\left(\frac{1}{\sqrt{2}}\right)$$

$$= \frac{1}{2\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{4}$$

Example 6: If $\cos\theta = -\frac{1}{3}$ and $\frac{\pi}{2} \le \theta \le \pi$, find the exact values for:

locates the angle



$$x^2 + y^2 = r^2$$
$$y^2 = 3^2 - (-1)^2$$
$$y^2 = 9 - 1$$
$$y = \sqrt{8}$$

a) $\sin 2\theta$

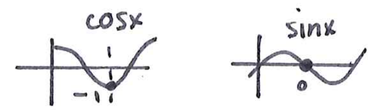
$$\sin 2\theta = 2\sin\theta\cos\theta$$
$$= 2\left(\frac{y}{r}\right)\left(\frac{x}{r}\right)$$
$$= 2\left(\frac{\sqrt{8}}{3}\right)\left(-\frac{1}{3}\right)$$
$$= -\frac{2\sqrt{8}}{9}$$

simplify

$$= -\frac{2\sqrt{4\cdot 2}}{9}$$
$$= -\frac{4\sqrt{2}}{9}$$

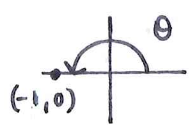
b) $\cos(\theta + \pi)$

A B



$$\cos(\theta + \pi) = \cos\theta\cos\pi - \sin\theta\sin\pi$$

evaluate



$$= \cos\theta(-1) - \sin\theta(0)$$
$$= \left(-\frac{1}{3}\right)(-1) - \left(\frac{\sqrt{8}}{3}\right)(0)$$
$$= \frac{1}{3}$$

Practice: p.306 #2, 4, 5, 8abc, 11, 12a, 16, 20